THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060B Mathematical Analysis II
Homework 5 Suggested Solutions

1. (Exercise 7.1.6 of [BS11])

   (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

   (b) Let $h(x) := 2$ if $0 \le x < 1$ and $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

Solution. (a) Let $\varepsilon > 0$ be given. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,2]$. Then consider $\dot{\mathcal{P}}_1$ the subset of $\dot{\mathcal{P}}$ having tags in $[0,1]$ and $\dot{\mathcal{P}}_2$ the subset of $\dot{\mathcal{P}}$ having tags in $[1,2]$. Then the union of the intervals in $\dot{\mathcal{P}}_1$ contains the interval $[0, 1 - \|\dot{\mathcal{P}}\|]$ and is contained in $[0, 1 + \|\dot{\mathcal{P}}\|]$. So we have that

$$2(1 - \|\dot{\mathcal{P}}\|) \le S(f; \dot{\mathcal{P}}_1) \le 2(1 + \|\dot{\mathcal{P}}\|).$$

Similarly, the union of the intervals in $\dot{\mathcal{P}}_2$ contains $[1 + \|\dot{\mathcal{P}}\|, 2]$ and is contained in $[1 - \|\dot{\mathcal{P}}_2\|, 2]$ and so we have

$$1 - \|\dot{\mathcal{P}}\| \le S(f; \dot{\mathcal{P}}_2) \le 1 + \|\dot{\mathcal{P}}\|.$$

Combining these two, we have that

$$3 - 3\|\dot{\mathcal{P}}\| \le S(f; \dot{\mathcal{P}}) = S(f; \dot{\mathcal{P}}_1) + S(f; \dot{\mathcal{P}}_2) \le 3 + 3\|\dot{\mathcal{P}}\| \Rightarrow \left| S(f; \dot{\mathcal{P}}) - 3 \right| \le 3\|\dot{\mathcal{P}}\|.$$

So taking $\|\dot{\mathcal{P}}\| < \dfrac{\varepsilon}{3}$ gives

$$\left| S(f; \dot{\mathcal{P}}) - 3 \right| \le 3\|\dot{\mathcal{P}}\| < \varepsilon$$

and hence we see that $f \in \mathcal{R}[0,2]$ and that $\int_0^2 f(x)dx = 3$.

(b) Let $\varepsilon > 0$ be given. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,2]$. Then consider $\dot{\mathcal{P}}_0$ the subset of $\dot{\mathcal{P}}$ having tags at 1. Then note that there are at most two subintervals in $\dot{\mathcal{P}}_0$ that contain 1 and moreover their union is contained in $[1 - \|\dot{\mathcal{P}}\|, 1 + \|\dot{\mathcal{P}}\|]$ and that this interval is of length $2\|\dot{\mathcal{P}}\|$, which means we have

$$\left| S(h; \dot{\mathcal{P}}_0) \right| \le 3 \cdot 2\|\dot{\mathcal{P}}\| = 6\|\dot{\mathcal{P}}\|.$$

Repeating the same arguments as in part (a) for $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$, we therefore obtain

$$3 - 3\|\dot{\mathcal{P}}\| - 6\|\dot{\mathcal{P}}\| \le S(h; \dot{\mathcal{P}}) = S(h; \dot{\mathcal{P}}_0) + S(h; \dot{\mathcal{P}}_1) + S(h; \dot{\mathcal{P}}_2) \le 3 + 3\|\dot{\mathcal{P}}\| + 6\|\dot{\mathcal{P}}\|$$
$$\Rightarrow \left| S(h; \dot{\mathcal{P}}) - 3 \right| \le 9\|\dot{\mathcal{P}}\|.$$

So taking $\|\dot{\mathcal{P}}\| < \dfrac{\varepsilon}{9}$ shows that $h \in \mathcal{R}[0,2]$ and that $\int_0^2 h(x)dx = 3$.

◀

2. (Exercise 7.1.8 of [BS11]) If $f \in \mathcal{R}[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, show that $\left|\int_a^b f\right| \le M(b-a)$.

   Solution. Since $f(x)$ is bounded from above by M, we have that $-M \le f(x) \le M$ for $x \in [a,b]$. Then we have that

$$\int_a^b -M \le \int_a^b f \le \int_a^b M \Rightarrow -M(b-a) \le \int_a^b f \le M(b-a).$$

   This then implies that

$$\left|\int_a^b f\right| \le M(b-a).$$

◀

3. (Exercise 7.1.10 of [BS11]) Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $\left(\dot{\mathcal{P}}_n\right)$ of tagged partitions of $[0,1]$ such that $\left\|\dot{\mathcal{P}}_n\right\| \to 0$ and $\lim_n S\left(g;\dot{\mathcal{P}}_n\right)$ exists.

   Solution. Since g is not bounded on $[0,1]$, it is not Riemann integrable. We now show the existence of a sequence of tagged partitions $\dot{\mathcal{P}}_n$ that satisfies the required condition. Define the partition of $[0,1]$ by

$$\mathcal{P}_n := \left\{x_0 = 0, x_1 = \frac{1}{n}, \ldots, x_{n-1} = \frac{n-1}{n}, x_n = 1\right\}$$

   and let $\dot{\mathcal{P}}_n$ be the partition $\mathcal{P}_n$ with tags t_k at the left end-points. Note that each of the tags t_k is rational. Then clearly we have that $\|\dot{\mathcal{P}}_n\| = \dfrac{1}{n} \to 0$ as $n \to +\infty$ but that we also have

$$S\left(g;\dot{\mathcal{P}}_n\right) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} 0 \cdot \frac{1}{n} = 0$$

   and hence $\lim\limits_{n\to+\infty} S(g;\dot{\mathcal{P}}_n) = 0$. ◀

4. (Exercise 7.1.15 of [BS11]) If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c,b+c]$ and that $\int\limits_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

   Solution. Let $\varepsilon > 0$ be given. Since f is Riemann integrable, there is a $\delta > 0$ such that for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$, we have

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon.$$

Now suppose $\dot{\mathcal{Q}} = \{([x_{k-1}, x_k], t_k)\}_{k=1}^n$ is a tagged partition of $[a+c, b+c]$ with $\|\dot{\mathcal{Q}}\| < \delta$. Define the tagged partition $\dot{\mathcal{P}}_Q$ by

$$\dot{\mathcal{P}}_Q = \{([x_{k-1} - c, x_k - c], t_k - c)\}_{k=1}^n .$$

Then since $x_{k-1} - c - (x_k - c) = x_{k-1} - x_k$ for any $k = 1, \ldots, n$, we can easily see that $\|\dot{\mathcal{P}}_Q\| = \|\dot{\mathcal{Q}}\| < \delta$. Moreover,

$$S(g; \dot{\mathcal{Q}}) = \sum_{k=1}^n g(t_k + c)(x_k - c - x_{k-1} + c) = \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) = S(f; \dot{\mathcal{P}}_Q)$$

and so

$$\left| S(g; \dot{\mathcal{Q}}) - \int_a^b f \right| = \left| S(f; \dot{\mathcal{P}}_Q) - \int_a^b f \right| < \varepsilon.$$

Hence, $g \in \mathcal{R}[a+c, b+c]$ with $\int_{a+c}^{b+c} g = \int_a^b f$. ◀

5. (Exercise 7.2.2 of [BS11]) Consider the function h defined by $h(x) := x + 1$ for $x \in [0, 1]$ rational, and $h(x) := 0$ for $x \in [0, 1]$ irrational. Show that h is not Riemann integrable.

Solution. Let $\varepsilon_0 = 1$ and let $\eta > 0$ be given. Then let $\mathcal{P}$ be any partition of $[0, 1]$ with $\|\mathcal{P}\| < \eta$. By the density of the rationals in $[0, 1]$ we can tag $\mathcal{P}$ with only rational tags (call this tagged partition $\dot{\mathcal{P}}$) and we have

$$\begin{aligned} S(h; \dot{\mathcal{P}}) &= \sum_{k=1}^n h(t_k)(x_k - x_{k-1}) = \sum_{k=1}^n (t_k + 1)(x_k - x_{k-1}) \\ &= \sum_{k=1}^n t_k(x_k - x_{k-1}) + \sum_{k=1}^n (x_k - x_{k-1}) = \sum_{k=1}^n t_k(x_k - x_{k-1}) + 1 \geq 1 \end{aligned}$$

since $t_k \geq 0$.

On the other hand, by the density of the irrationals in $[0, 1]$, then we can tag $\mathcal{P}$ with only irrational tags (call this tagged partition $\dot{\mathcal{Q}}$) and we have

$$S(h; \dot{\mathcal{Q}}) = \sum_{k=1}^n h(t_k)(x_k - x_{k-1}) = \sum_{k=1}^n 0 \cdot (x_k - x_{k-1}) = 0.$$

Hence, we have that

$$\left| S(h; \dot{\mathcal{P}}) - S(h; \dot{\mathcal{Q}}) \right| = 1 = \varepsilon_0$$

which is true for arbitrarily small $\eta > 0$. Hence by the negation of the Cauchy criterion (7.2.1 of [BS11]), we see that h is not Riemann integrable. ◀

6. (Exercise 7.2.3 of [BS11]) Let $H(x) := k$ for $x = 1/k$ ($k \in \mathbb{N}$) and $H(x) := 0$ elsewhere on $[0, 1]$. Use Exercise 1, or the argument in 7.2.2(b), to show that H is not Riemann integrable.

Solution. Let $\varepsilon_0 = \frac{1}{2}$ and let $\eta > 0$ be given. By the Archimedean property, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $\dfrac{1}{n} < \eta$. Consider the partition

$$\mathcal{P} = \left\{ x_0 = 0, x_1 = \frac{1}{n}, \ldots, x_{n-1} = \frac{n-1}{n}, x_n = 1 \right\}.$$

Let $\dot{\mathcal{P}}$ be the tagged partition consisting of $\mathcal{P}$ with the tags at the left end-points and let $\dot{\mathcal{Q}}$ be the tagged partition consisting of $\mathcal{P}$ with the tags at irrational points (which we can always take because of the density of the irrationals in $[0,1]$). Then we have that

$$S(h; \dot{\mathcal{P}}) = \sum_{k=1}^{n} h\left(\frac{1}{k}\right)\left(\frac{k}{n} - \frac{k-1}{n}\right) = \sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2} \geq \frac{1}{2} \quad \text{for any } n \geq \max\{N, 1\}.$$

On the other hand, we have

$$S(h; \dot{\mathcal{Q}}) = \sum_{k=1}^{n} h(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0.$$

Hence, we have

$$\left| S(h; \dot{\mathcal{P}}) - S(h; \dot{\mathcal{Q}}) \right| \geq \frac{1}{2} = \varepsilon_0$$

and so h is not Riemann integrable by the Cauchy criterion. ◀

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition.* Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.